

On de Broglie's soliton wave function of many particles with finite masses, energies and momenta

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Abstract

We consider a mass-less manifestly covariant *linear* Schrödinger equation. First, we show that it possesses a class of non-dispersive soliton solution with finite-size spatio-temporal support inside which the quantum amplitude satisfies the Klein-Gordon equation with finite *emergent* mass. We then proceed to interpret the soliton wave function as describing a particle with finite mass, energy and momentum. Inside the spatio-temporal support, the wave function shows spatio-temporal internal vibration with angular frequency and wave number that are determined by the energy-momentum of the particle as firstly conjectured by de Broglie. Imposing resonance of the internal vibration inside the spatio-temporal support leads to Planck-Einstein quantization of energy-momentum. The first resonance mode is shown to recover the classical energy-momentum relation developed in special relativity. We further show that the linearity of the Schrödinger equation allows one to construct many solitons solution through superposition, each describing a particle with various masses, energies and momenta.

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I. INTRODUCTION: RELATIVISTIC MADELUNG FLUID

Surprisingly, despite of the remarkable pragmatism of quantum theory, there is still an unsettled debate concerning the physical status of its main ingredient, that is the wave function. Roughly speaking, there are two main attitudes toward this foundational issue [1, 2, 3, 4].

The first cult considers that the wave function has no physical reality at all. They assume the wave function as the representation of *our knowledge* about the physical reality rather than to refer to the reality itself. This interpretation then regards quantum theory as a theory concerning our knowledge about the reality obtained through experiment rather than a theory about physical reality. It is developed by Bohr and Heisenberg and commonly known as Copenhagen interpretation. This line of thought eventually led to the *probabilistic view* of wave function through Born's rule [5].

The second attitude is to consider the wave function as a real *physical field* referring directly to the physical object being described, like say electromagnetic field. There are many interpretation of quantum theory which attribute such a physical status to the wave function. The mostly mentioned interpretations which support this view includes the axiomatic-most-“used” standard interpretation of Dirac-von Neumann [6, 7], the Bohmian mechanics [4, 8], many worlds interpretation [9, 10], the theory of spontaneous localization [11], etc.

The less mentioned one is de Broglie's theory of double solutions [12, 13]. In his attempt to solve the dual nature of matter as particle and wave, he was searching for a *nonlinear wave equation* which assumes a non-dispersive soliton solution at the amplitude which is sufficiently high, while possesses a linear solution satisfying a linear superposition at the amplitude which is weak. He then proposed that the soliton part should be regarded as a particle which is guided by the linear part of the solution. This idea eventually led him to derive his famous guiding principle:

$$E = \hbar\omega, \quad \mathbf{p} = \hbar\mathbf{k}, \quad (1)$$

which relates the energy-momentum $\{E, \mathbf{p}\}$ of the particle with the angular frequency-wave number $\{\omega, \mathbf{k}\}$ of the linear wave.

Equations (1) can be argued as the most important principle of quantum mechanics [14]. Partly inspired by those relations, Schrödinger developed his celebrated equation. Yet in

contrast to de Broglie's original idea, Schrödinger equation is linear with respect to the wave function. In this theory, a free particle is then usually represented by a plane wave satisfying the relations of Eqs. (1). Despite physically vague, the plane wave representation surprisingly works for all pragmatical purposes [3]. Yet one can argue that the successes of this representation relies heavily on the above de Broglie's relation [13]. It is apparently the pragmatical successes of the linear Schrödinger equation if combined with the Born's rule that eventually discourage people from further continuing de Broglie's program [15, 16, 17, 18, 19]. In this paper, by considering a mass-less relativistic *linear* Schrödinger equation, we shall show that it has a new class of soliton solutions with properties exactly envisioned long time ago by de Broglie.

Let us consider a closed system whose state is uniquely determined by a complex-valued wave function in spacetime, $\psi(q)$, where $q = (q^0, q^1, q^2, q^3) = (ct, x, y, z)$. Further, let us assume that the wave function satisfies the following manifestly covariant mass-less Schrödinger equation

$$i\hbar \frac{\partial}{\partial \lambda} \psi(q; \lambda) = \frac{\hbar^2}{2} \square \psi(q; \lambda). \quad (2)$$

Here, λ is some affine parameter, $\square = -\eta^{ab} \partial_a \partial_b$ and $\eta^{ab} = \text{diag}(-1, 1, 1, 1)$ are D'Alembertian operator and flat Minkowskian metric, respectively. Eq. (2) has been proposed to interpret the Klein-Gordon equation through a particle model [20, 21].

Next, putting the wave function into polar form, $\psi = I \exp(iS/\hbar)$, where I and S are real-valued functions, and separating into the real and imaginary parts, one obtains [4]:

$$\frac{dv^a}{d\lambda} = -\partial^a U, \quad \frac{\partial \rho}{\partial \lambda} + \partial_a (\rho v^a) = 0. \quad (3)$$

Here, $\rho(q; \lambda) = |\psi|^2 = I^2$ is the quantum probability density, $v^a(q; \lambda)$ is a velocity field generated by the quantum phase $S(q; \lambda)$ as

$$v^a(q; \lambda) = \partial^a S(q; \lambda), \quad (4)$$

and $U(q; \lambda)$ is generated by the quantum amplitude $I(q; \lambda)$ as

$$U(q) = \frac{\hbar^2}{2} \frac{\square I}{I}. \quad (5)$$

We have thus adopted the Madelung fluid picture for the Schrödinger equation [22]. Due to its formal similarity with the Euler equation in hydrodynamics, the term on the right hand

side of the left equation in Eqs. (3), $F^a = -\partial^a U$, is called as quantum force field. Thus, correspondingly, $U(q)$ is called as quantum potential.

Let us remark that written in the form of Madelung fluid, it becomes clear that the original Schrödinger equation possesses a hidden self-referential property. Namely, the quantum potential $U(q)$ is generated by the quantum probability density $\rho(q)$ through Eq. (5). This in turn will dictate the way $\rho(q)$ must evolve with time through Eqs. (3) and so on and so forth. It is thus reasonable to expect some interesting *self-organized* physically relevant phenomena. In particular, in this paper we shall be interested to study the *fixed points* of such dynamics.

II. SELF-TRAPPED QUANTUM PROBABILITY DENSITY

Let us proceed to specify a class of wave functions whose quantum probability density is further related to its own quantum potential as [23]:

$$\rho(q) = \frac{1}{Z(T)} \exp\left(-\frac{U(q)}{T}\right), \quad (6)$$

where T is a real-valued parameter below chosen to be non-negative and $Z(T)$ is a normalization factor. We shall show that the above class of quantum probability densities possesses non-trivial and physically interesting properties. To do this, notice that combined with the definition of quantum potential given in Eq. (5), Eq. (6) comprises a differential equation for $U(q)$ or $\rho(q)$ subjected to the condition that $\rho(q)$ must be normalized. In term of $U(q)$, one has to solve the following nonlinear differential equation [23]:

$$-\square U = \frac{1}{2T} \partial^a U \partial_a U + \frac{4T}{\hbar^2} U. \quad (7)$$

One observes that the above differential equation is invariant under Lorentz transformation. Hence, given a solution $U(q)$, then any function $U(q')$, where $q'^a = \Lambda_b^a q^b$ and Λ_b^a is Lorentz transformation, is also a solution of Eq. (7).

Let us develop a class of solutions in which the quantum probability density is being trapped by the quantum potential it itself generates [23]. To do this, let us assume that there is an inertial frame so that the quantum probability density is separable into its spatial and temporal parts as follows:

$$\rho(q) = \rho_{\mathbf{x}}(\mathbf{x}) \rho_t(t), \quad \mathbf{x} = \{x, y, z\}. \quad (8)$$

In this case, the quantum potential can then be decomposed into

$$U(q) = U_{\mathbf{x}}(\mathbf{x}) + U_t(t), \quad (9)$$

where

$$U_{\mathbf{x}}(\mathbf{x}) = -\frac{\hbar^2}{2} \frac{\partial_{\mathbf{x}}^2 I_{\mathbf{x}}}{I_{\mathbf{x}}}, \quad U_t(t) = \frac{\hbar^2}{2c^2} \frac{\partial_t^2 I_t}{I_t}. \quad (10)$$

Here, $I_i \equiv \sqrt{\rho_i}$ with $i = \mathbf{x}, t$; and $\partial_{\mathbf{x}}^2 \equiv \partial_{\mathbf{x}} \cdot \partial_{\mathbf{x}}$ where $\partial_{\mathbf{x}} = \{\partial_x, \partial_y, \partial_z\}$.

The condition of Eq. (8) is not Lorentz invariant so is the resulting class of solutions we are going to develop. Yet, its nontrivial property will be shown to be Lorentz invariant. Inserting Eq. (9), Eq. (7) can thus be re-collected as $\partial_{\mathbf{x}}^2 U_{\mathbf{x}} - (1/2T) \partial_{\mathbf{x}} U_{\mathbf{x}} \cdot \partial_{\mathbf{x}} U_{\mathbf{x}} - (4T/\hbar^2) U_{\mathbf{x}} = (1/c^2) \partial_t^2 U_t - (1/2T) (\partial_t U_t)^2 + (4T/\hbar^2) U_t = D$, where D is constant. Below for simplicity we shall take the case when $D = 0$. One thus has to solve the following decoupled pair of nonlinear differential equations:

$$\begin{aligned} \partial_{\mathbf{x}}^2 U_{\mathbf{x}} - \frac{1}{2T} \partial_{\mathbf{x}} U_{\mathbf{x}} \cdot \partial_{\mathbf{x}} U_{\mathbf{x}} - \frac{4T}{\hbar^2} U_{\mathbf{x}} &= 0, \\ \frac{1}{c^2} \partial_t^2 U_t - \frac{1}{2T} (\partial_t U_t)^2 + \frac{4T}{\hbar^2} U_t &= 0. \end{aligned} \quad (11)$$

A. Spatial self-trapping

Let us first discuss the spatial part by solving the upper differential equation in Eqs. (11). To do this, let us search for a class of solutions in which the spatial quantum probability density is further separable as

$$\rho_{\mathbf{x}}(\mathbf{x}) = \rho_x(x) \rho_y(y) \rho_z(z), \quad (12)$$

so that the spatial part of the quantum potential is further decomposable into

$$U_{\mathbf{x}}(\mathbf{x}) = U_x(x) + U_y(y) + U_z(z), \quad U_i(i) = -\frac{\hbar^2}{2} \frac{\partial_i^2 I_i}{I_i}, \quad (13)$$

where $i = x, y, z$. Putting this ansatz into the upper differential equation in Eqs. (11), one can choose a class of solutions in which each $U_i(i)$ satisfies the following decoupled nonlinear differential equations:

$$\partial_i^2 U_i = \frac{1}{2T} (\partial_i U_i)^2 + \frac{4T}{\hbar^2} U_i, \quad i = x, y, z. \quad (14)$$

To avoid complicated notation, below we shall consider only the x -degree of freedom. The other two spatial degrees of freedom $\{y, z\}$ follows similarly.

Figure 1a shows the numerical solutions of Eq. (14) with the boundary conditions: $\partial_x U_x(0) = 0$ and $U_x(0) = 1$, for several small values of T . The reason for choosing small T will be clear later. One can see that the spatial part of quantum probability density $\rho_x(x)$ is being trapped by its own self-generated quantum potential $U_x(x)$ [23]. Moreover, there is a finite distance $x = \pm x_m$ at which the partial quantum potential is blowing-up $U_x(\pm x_m) = \infty$, so that the corresponding partial quantum probability density is vanishing, $\rho_x(\pm x_m) = 0$. This is a familiar phenomena in nonlinear differential equation [24], which for the case at hand, can be proven as follows.

Let us define a new variable $u_x(x) = \partial_x U_x$. The nonlinear differential equation of Eq. (14) then transforms into

$$\partial_x u_x = \frac{1}{2T} u_x^2 + \frac{4T}{\hbar^2} U_x. \quad (15)$$

The boundary condition translates into $u_x(0) = \partial_x U_x(0) = 0$. Further, let us now consider the following nonlinear differential equation

$$\partial_x \tilde{u}_x = \frac{1}{2T} \tilde{u}_x^2 + \frac{4T}{\hbar^2} X, \quad (16)$$

where $X \equiv U_x(0)$; with $\tilde{u}_x(0) = 0$. Since $U_x(x) \geq U_x(0) = X$, then it is obvious that $|u_x(x)| \geq |\tilde{u}_x(x)|$.

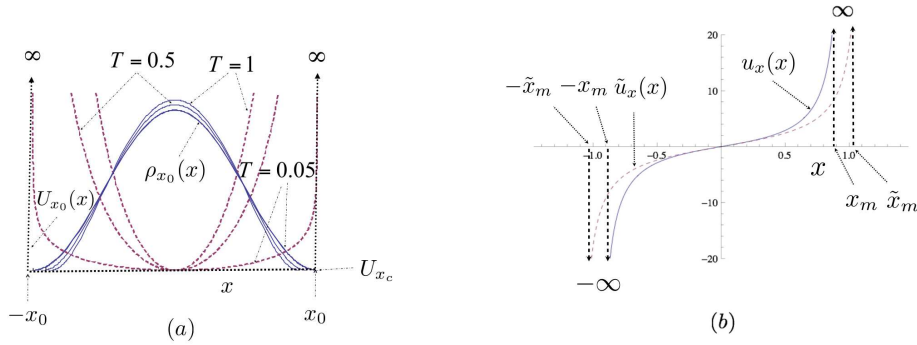


FIG. 1: (a) The x -part of spatial quantum probability density $\rho_x(x)$ (solid line) and its corresponding partial quantum potential (dashed line) $U_x(x)$ for several small values of T obtained by solving equation (14). We also plot the analytical solution for $\rho_{x_0}(x)$ at $T = 0$, assuming that $U_{x_0}(x)$ takes the form of a box with infinite wall. (b) $u_x(x)$ and $\tilde{u}_x(x)$. See text for detail.

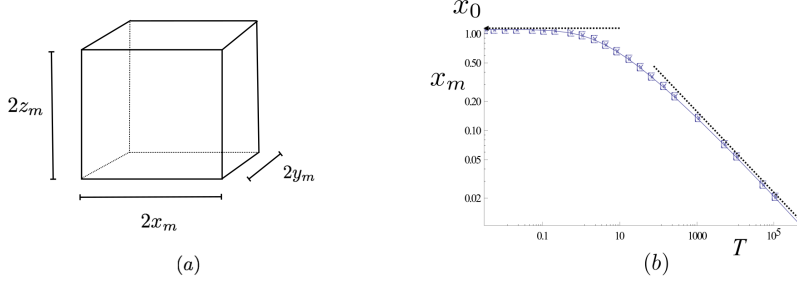


FIG. 2: (a) The square box spatial support and (b) the half-length of the x -side of the box x_m plotted against T .

One can then solve the latter nonlinear differential equation of Eq. (16) analytically to have:

$$\tilde{u}_x(x) = a \tan(bx), \quad a = \frac{2T}{\hbar} \sqrt{2X}, \quad b = \frac{1}{\hbar} \sqrt{2X}. \quad (17)$$

It is then clear that at $x = \pm \tilde{x}_m = \pm \pi/(2b)$, \tilde{u}_x is blowing-up, namely $\tilde{u}_x(\pm \tilde{x}_m) = \pm \infty$. Recalling the fact that $|u_x(x)| \geq |\tilde{u}_x(x)|$, then $u_x(x)$ is also blowing-up at points $x = \pm x_m$, $u_x(\pm x_m) = \pm \infty$, where $x_m \leq \tilde{x}_m$. See Fig. 1b. Hence, one can conclude that $U_x(x)$ is also blowing-up at $x = \pm x_m$, $U_x(\pm x_m) = \infty$. It is then safe to say that the x -part of spatial self-trapped quantum probability density $\rho_x(x)$ possesses only a finite range of spatial support: $\mathcal{M}_x = [-x_m, x_m]$. Finally, the spatial part self-trapped quantum probability density $\rho_{\mathbf{x}}(\mathbf{x})$ possesses only finite-size spatial support: $\mathcal{M}_{\mathbf{x}} = [-x_m, x_m] \otimes [-y_m, y_m] \otimes [-z_m, z_m]$, which takes the form of a three dimensional square box with sides length $2i_m$, $i = x, y, z$. See Fig. 2a.

Let us see what happens if one varies the parameter T . Figure 2b shows the values of x_m as a function of T obtained by numerically solving the differential equation of Eq. (14) with fixed boundary conditions: $\partial_x U_x(0) = 0$, $U_x(0) = 1$. One can see that as we increase T , x_m decreases, and eventually vanishes for infinite value of T . This shows that $\rho_x(x)$ is converging toward a delta function for infinite T . A very interesting fact is seen for the opposite limit of vanishing T . One observes that $\lim_{T \rightarrow 0} x_m(T) = x_0$, where x_0 is finite. This fact suggests to us that at $T = 0$, the spatial quantum probability density and thus its corresponding quantum potential are converging toward certain functions:

$$\lim_{T \rightarrow 0} \rho_x(x; T) = \rho_{x_0}(x), \quad \lim_{T \rightarrow 0} U_x(x; T) = U_{x_0}(x). \quad (18)$$

Let us discuss this latter asymptotic situation in more detail. First, one can see in Fig. 1a that as T decreases, the quantum potential inside the spatial support is getting flatter before becoming infinite at the boundary points, $x = \pm x_m(T)$. One might then guess that at $T = 0$, the quantum potential is perfectly flat inside the one dimensional box of spatial support and is infinite at its boundary points: $x = \pm x_0$. Guided by this guess, let us calculate the profile of the spatial quantum probability density for vanishing value of T . To do this, let us denote the assumed positive definite constant value of the quantum potential inside the support as U_{x_c} . Recalling the definition of spatial quantum potential $U_x(x)$ given in Eq. (13) one has

$$\partial_x^2 I_{x_0}(x) = -\frac{2U_{x_c}}{\hbar^2} I_{x_0}(x), \quad (19)$$

where $I_{x_0} \equiv \sqrt{\rho_{x_0}}$. The above differential equation must be subjected to the spatial boundary condition: $\rho_{x_0}(\pm x_0) = I_{x_0}^2(\pm x_0) = 0$. Solving Eq. (19) one has

$$I_{x_0}(x) = A_{x_0} \cos(k_{x_0}x), \quad (20)$$

where A_{x_0} is normalization constant and the wave number k_{x_0} is related to the quantum potential as:

$$k_{x_0} = \sqrt{2U_{x_c}/\hbar^2}. \quad (21)$$

The boundary condition implies

$$k_{x_0}x_0 = \pi/2. \quad (22)$$

Figure 1a shows that as T decreases toward zero, $\rho_x(x; T)$ obtained by numerically solving Eq. (14) is indeed converging toward $\rho_{x_0}(x)$ obtained in Eq. (20). This observation thus confirms our guess that *at $T = 0$, the x -part spatial quantum potential is flat inside the spatial support \mathcal{M}_x and is infinite at its boundary points.*

Hence, at $T = 0$, in total the spatial part of the quantum probability density can be written as

$$\rho_{\mathbf{x}_0}(\mathbf{x}) = \prod_{i=x,y,z} \rho_{i_0}(i), \quad (23)$$

where $\rho_{i_0} = A_{i_0} \cos(k_{i_0}i)$, $i = x, y, z$. Moreover, the support is given by three dimensional volume $\mathcal{M}_{\mathbf{x}_0} = [-x_0, x_0] \otimes [-y_0, y_0] \otimes [-z_0, z_0]$ at surface of which the spatial quantum potential is blowing-up so that the quantum probability density is vanishing.

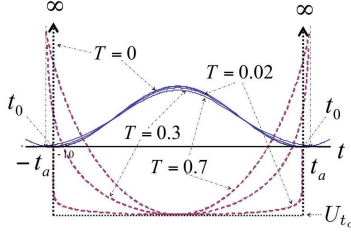


FIG. 3: The profile of temporal self-trapped quantum probability density $\rho_t(t)$ (solid line) and its corresponding temporal quantum potential $U_t(t)$ (dashed line) for several small values of T . See text for detail.

B. Temporal self-trapping

Next, let us discuss the temporal part of quantum probability density $\rho_t(t)$ by solving the lower differential equation in Eqs. (11). In particular, we are interested to investigate the behavior of $\rho_t(t)$ at the limit $T \rightarrow 0$, if it exists. Figure 3 shows the solution with the boundary: $\partial_t U_t(0) = 0$ and $U_t(0) \equiv U_{t_c} = -1$. $\rho_t(t)$ and the corresponding $U_t(t)$ are plotted for several small values of T . One can again see similar phenomena with the spatial part that $\rho_t(t)$ is being self-trapped by the corresponding $U_t(t)$. One also sees that the support of $\rho_t(t)$ is finite given by the interval $\mathcal{M}_t = [-t_a, t_a]$ at the boundary points of which the temporal quantum potential is blowing-up: $U_t(\pm t_a) = \infty$.

In Fig. 4 we plot the variation of t_a against T while keeping $U_t(0) \equiv U_{t_c}$ fixed. First, in contrast to the spatial part, one observes that t_a is a monotonically increasing function of T . Further, in contrast to the spatial part in which the length of the support is vanishing for infinite value of T , the length of the support of temporal quantum probability density $\rho_t(t)$ is blowing-up at finite value of $T = T_p(U_{t_c})$. See Fig. 4. Given U_{t_c} , then for $T \geq T_p(U_{t_c})$, the temporal quantum potential $U_t(t)$ is no more convex everywhere, but is almost periodic as depicted in Fig. 5. Hence, $\rho_t(t)$ as defined in Eq. (6) is no more normalizable. This case is therefore physically irrelevant.

Yet, again, as in the case of spatial quantum probability density, as one decreases T

toward zero, t_a is converging toward a finite value t_0 :

$$\lim_{T \rightarrow 0} t_a \equiv t_0. \quad (24)$$

This again shows that at $T = 0$, $U_t(t)$ and $\rho_t(t)$ will converge toward some functions:

$$\lim_{T \rightarrow 0} U_t(t; T) = U_{t_0}(t), \quad \lim_{T \rightarrow 0} \rho_t(t; T) \equiv \rho_{t_0}(t). \quad (25)$$

Below we shall be interested to further study the case of vanishing T .

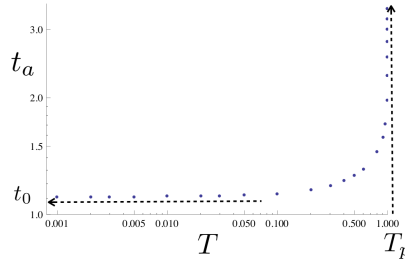


FIG. 4: The variation of half support t_a of the self-trapped temporal quantum probability density $\rho_t(t)$ against T . See text for detail.

Proceeding in the same way as for the spatial part, let us calculate $\rho_{t_0}(t)$. To do this, first one observes in Fig. 3 that as T is approaching zero, $U_t(t)$ is getting flatter inside the support before becoming infinite at the boundary points, $t = \pm t_a(T)$. Again, let us guess that at $T = 0$, the temporal quantum potential is perfectly flat inside the support $\mathcal{M}_{t_0} = (-t_0, t_0)$, given by $U_{t_c} < 0$; and is infinite at $t = \pm t_0$. Recalling the definition of $U_t(t)$ given in Eq. (10), one has

$$\partial_t^2 I_{t_0} = \frac{2c^2 U_{t_c}}{\hbar^2} I_{t_0}. \quad (26)$$

Here $I_{t_0} \equiv \sqrt{\rho_{t_0}}$. The above differential equation must be subjected to the boundary condition: $I_{t_0}(\pm t_0) = 0$. Solving Eq. (26), one obtains:

$$I_{t_0}(t) = A_{t_0} \cos(\omega_0 t). \quad (27)$$

Here A_{t_0} is a normalization constant and the angular frequency ω_0 is related to the quantum potential as

$$\omega_0 = \sqrt{(-2c^2 U_{t_c} / \hbar^2)}. \quad (28)$$

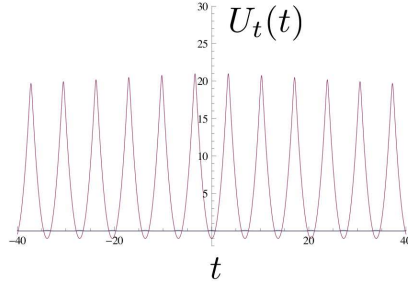


FIG. 5: Periodic temporal quantum potential for the case $T = 1$ with $\partial_t U_t(0) = 0$ and $U_t(0) = -1$. See text for detail.

The boundary imposes:

$$\omega_0 t_0 = \pi/2. \quad (29)$$

One finally sees in Fig. 3 that as one decreases T toward zero, $\rho_t(t; T)$ obtained by solving the lower differential equation in Eqs. (11) is indeed converging toward $\rho_{t_0}(t)$ given in Eq. (27). *This again justifies our guess that at $T = 0$, $U_t(t)$ is perfectly flat inside \mathcal{M}_{t_0} , and is infinite at $t = \pm t_0$.*

III. SPACETIME SOLITON

Hence, in total, at $T = 0$, $\rho_0(q) \equiv \rho_{\mathbf{x}_0}(\mathbf{x})\rho_{t_0}(t)$ satisfies the differential equation of Eq. (7). Notice that the spatio-temporal support of $\rho_0(q)$ is composed by $\mathcal{M} \equiv \mathcal{M}_{\mathbf{x}_0} \otimes \mathcal{M}_{t_0}$. Inside \mathcal{M} , the quantum potential is thus flat given by

$$U(q) = U_{x_c} + U_{y_c} + U_{z_c} + U_{t_c} = \frac{1}{2} \left(\hbar^2 k_{\mathbf{x}_0}^2 - \frac{\hbar^2 \omega_0^2}{c^2} \right) \equiv U_0, \quad (30)$$

where $k_{\mathbf{x}_0}^2 = \mathbf{k}_{\mathbf{x}_0} \cdot \mathbf{k}_{\mathbf{x}_0}$ and $\mathbf{k}_{\mathbf{x}_0} \equiv \{k_{x_0}, k_{y_0}, k_{z_0}\}$; and we have employed Eqs. (21) and (28). One therefore observes that at $T = 0$, the quantum force is vanishing inside the spatio-temporal support, $\partial^a U = 0$.

Now, at $\tau = 0$, let us choose the following initial wave function, $\{\rho_0(q), v_0^a(q)\}$. Here $\rho_0(q) = \rho_{\mathbf{x}_0}(\mathbf{x})\rho_{t_0}(t)$ and $v_0^a(q)$ is a uniform velocity vector field having non-vanishing value

only inside the spatio-temporal support \mathcal{M} , that is $v_0^a(q) = v_C^a$, where v_C^a is a constant four velocity vector. Since at $\tau = 0$ the quantum force is vanishing, then initially one has $dv^a/d\tau = 0$. Hence, at infinitesimal lapse of proper time, $\tau = \Delta\tau$, the velocity field is kept uniform and constant. This in turn will shift the initial quantum probability density in spacetime by $\Delta q^a = v_C^a \Delta\tau$, while keeping its profile unchanged: $\rho(q; \Delta\tau) = \rho_0(q^a - v_C^a \Delta\tau)$. Accordingly, the spatio-temporal support will also be shifted by the same amount: $\mathcal{M}_{\Delta\tau}$. The same thing will happen for the next infinitesimal lapse of proper time and so on and so forth. Hence, at finite lapse of proper time τ , one concludes that the pair of fields

$$\{\rho(q; \tau), v^a(q; \tau)\} = \{\rho_0(q^b - v_C^b \tau), v_0^a(q^b - v_C^b \tau)\}, \quad (31)$$

comprises the stationary wave function of the relativistic Madelung fluid dynamics. Here, q belongs to the spatio-temporal support at proper time τ , denoted by \mathcal{M}_τ . *We have thus a spatio-temporally localized wave packet traveling in spacetime: namely a spacetime soliton.* See Fig. 6.

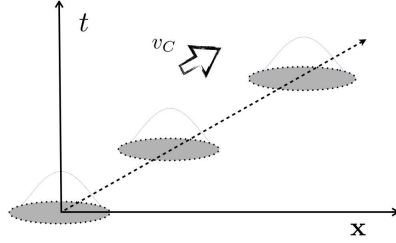


FIG. 6: Spatio-temporally localized wave packet moving in spacetime.

A. Mass

Next, before proceeding to write down the explicit form of the spacetime soliton wave function, let us first discuss the physical meaning of quantum potential. First, since the quantum potential is constant inside \mathcal{M}_τ given by U_0 , one has $\bar{U} = \int dq U \rho = U_0$. Let us proceed to choose a sufficiently large ω_0 by picking sufficiently large $|U_{t_c}|$ so that $\bar{U} = U_0$

given in Eq. (30) is negative, $\bar{U} < 0$. This allows us to define a new quantity m as

$$\bar{U} = U_0 = -\frac{1}{2}m^2c^2, \quad (32)$$

so that inserting into Eq. (30) one obtains

$$\frac{\hbar^2\omega_0^2}{c^2} - \hbar^2k_{\mathbf{x}0}^2 = m^2c^2. \quad (33)$$

Recalling again the definition of quantum potential of Eq. (5), Eq. (32) can be rewritten as

$$\square I(q; \tau) + \frac{m^2c^2}{\hbar^2}I(q; \tau) = 0, \quad q \in \mathcal{M}_\tau, \quad (34)$$

where $I \equiv \sqrt{\rho}$.

A physical interpretation to the above formalism is in order. Eq. (34) is but the Klein-Gordon equation with mass term m . One however should keep in mind that the differential equation of Eq. (34) must be subjected to the boundary condition that $I(q; \tau)$ is vanishing at the surface boundary of the spatio-temporal support, \mathcal{M}_τ . One can also see that though $I(q)$ is not Lorentz invariant, m is. Moreover, since U_0 is conserved so is m .

B. Energy-momentum

Let us proceed to discuss another conserved quantity: $v^a = v_C^a$. To do this, let us use the conserved invariant quantity m developed in the previous subsection to rescale the affine parameter λ as follows:

$$\tilde{\lambda} = m\lambda. \quad (35)$$

One thus has $\partial/\partial\lambda = m\partial/\partial\tilde{\lambda}$. Using this, Eq. (2) becomes

$$i\hbar\frac{\partial\psi}{\partial\tilde{\lambda}} = \frac{\hbar^2}{2m}\square\psi. \quad (36)$$

Moreover, the pair of equation in Eqs. (3) translates into

$$m\frac{d\tilde{v}^a}{d\tilde{\lambda}} = -\partial^a\tilde{U}, \quad \frac{\partial\rho}{\partial\tilde{\lambda}} + \partial_a(\rho\tilde{v}^a) = 0, \quad (37)$$

where $\tilde{v}^a \equiv dq^a/d\tilde{\lambda}$, and is now related to the quantum phase $S(q)$ as

$$\tilde{v}^a = \partial^a S/m = v^a/m, \quad (38)$$

and \tilde{U} is given by

$$\tilde{U} = U/m = \frac{\hbar^2}{2m} \frac{\square I}{I}. \quad (39)$$

Let us then use the proper time τ for the affine parameter, $\tau = \tilde{\lambda}$, and to avoid complicated notation let us rewrite the rescaled four velocity vector \tilde{v}^a back as v^a . One can then define four momentum vector as

$$p^a = mv^a. \quad (40)$$

which is conserved. The spatial part of the above quantity gives us the usual definition of classical momentum:

$$\mathbf{p} = m\mathbf{v}. \quad (41)$$

From the temporal part one can define a scalar quantity E as

$$E = cp^0 = \frac{mc^2}{\sqrt{1 - (v/c)^2}} \rightarrow mc^2 + \frac{1}{2}mv^2, \quad (42)$$

where v is the absolute value of spatial velocity vector, $v = \|\mathbf{v}\|$, and the arrow implies that v/c is sufficiently small. Hence, E has the dimension of energy. The second term is the kinetic energy and the first term is usually dubbed as rest-mass energy, namely the energy when the soliton is not moving.

Next, let us calculate a quantity which is convenient for later discussion defined as follows:

$$\langle P \rangle = \int_{\mathcal{M}} dq \psi^*(q) \left(\frac{\hbar^2}{2m} \square \right) \psi(q). \quad (43)$$

Since it is conserved, then it is sufficient to calculate its value at $\tau = 0$. Putting the wave function in polar form $\psi = I_0 \exp(iS_0/\hbar)$, one gets

$$\begin{aligned} \langle P \rangle = \int_{\mathcal{M}} dq \left(I_0 \frac{\hbar^2}{2m} \square I_0 + \frac{1}{2m} I_0^2 \partial_a S_0 \partial^a S_0 \right. \\ \left. - \frac{i\hbar}{m} I_0 \partial_a I_0 \partial^a S_0 + i \frac{\hbar^2}{2m} I_0^2 \square S_0 \right). \end{aligned} \quad (44)$$

The first term on the right hand side is nothing but the average quantum potential divided by mass m :

$$\int_{\mathcal{M}} dq I_0 \frac{\hbar^2}{2m} \square I_0 = \frac{1}{m} \int_{\mathcal{M}} dq \rho_0(q) U_0(q) = \frac{\bar{U}_0}{m}. \quad (45)$$

Recalling the relation between the four velocity vector field given in Eq. (38), the second term on the right hand side is given by

$$\bar{K}_0 \equiv \int_{\mathcal{M}} dq \rho_0(p_a p^a / (2m)) = \frac{p_a p^a}{2m}, \quad (46)$$

where in the last equality we have used the fact that the velocity vector is uniform inside the spatio-temporal support. Further, for the case of soliton where $v^a = \partial^a S_0/m$ is uniform, the last term is vanishing due to the vanishing divergence $\partial_a v^a = -\square S_0/m = 0$. Finally, the third term is also vanishing due to the fact that $I_0(q) = I_{t_0}(t)I_{x_0}(x)I_{y_0}(y)I_{z_0}(z)$ is separable and each $I_{i_0}(i)$ possesses a symmetry $I_{i_0}(i) = I_{i_0}(-i)$ so that $\partial_i I_{i_0}$ satisfies $\partial_i I_{i_0}(-i) = -\partial_i I_{i_0}(i)$, $i = t, x, y, z$. Hence, in total one obtains

$$\begin{aligned}\langle P \rangle &= \frac{\bar{U}_0}{m} + \bar{K}_0 = \frac{1}{2m} \left(\hbar^2 k_{\mathbf{x}_0}^2 - \frac{\hbar^2 \omega_0^2}{c^2} + p_a p^a \right) \\ &= \frac{(-m^2 c^2 + p^a p_a)}{2m} = -mc^2.\end{aligned}\tag{47}$$

Hence it is given by the negative of the rest-mass energy.

C. de Broglie's wave function for a single free particle

Now let us write down the complex-valued spacetime wave function $\psi(q; \tau) = I \exp(iS/\hbar)$ corresponding to the spacetime soliton that we have just developed. To do this, one has to calculate the quantum phase by integrating $\partial^a S = p^a$ to give us

$$S(q; \tau) = -\frac{E}{c}(ct) + p_x x + p_y y + p_z z + \xi(\tau),\tag{48}$$

where $\xi(\tau)$ is a function only of τ . Hence, at proper time τ , one gets

$$\begin{aligned}\psi(q; \tau) &= I_{\mathbf{x}_0}(\mathbf{x} - \mathbf{v}\tau) I_{t_0}(ct - E\tau/(mc)) \\ &\quad \times \exp \left(i(-Et + p_x x + p_y y + p_z z + \xi(\tau))/\hbar \right).\end{aligned}\tag{49}$$

Inserting this into the relativistic Schrödinger equation of Eq. (2) one can show that $\xi(\tau)$ is related to $\langle P \rangle$ as $d\xi/d\tau = -\langle P \rangle$ which can be integrated to give $\xi(\tau) = -\langle P \rangle \tau = mc^2 \tau$ up to some constant. Putting this back into Eq. (49) one finally obtains

$$\begin{aligned}\psi(q; \tau) &= I_{\mathbf{x}_0}(\mathbf{x} - \mathbf{v}\tau) I_{t_0}(ct - E\tau/(mc)) \\ &\quad \times \exp \left(i(-Et + p_x x + p_y y + p_z z + mc^2 \tau)/\hbar \right).\end{aligned}\tag{50}$$

One concludes that the phase of the soliton wave function uniquely gives the mass-energy-momentum of the particle.

One can also see from Eq. (50) that the soliton wave function possesses internal vibration in spacetime whose wave number and angular frequency are given by the four momentum vector p^a as

$$k_{Bi} = p_i/\hbar, \quad i = x, y, z, \quad \text{and} \quad \omega_B = E/\hbar. \quad (51)$$

The above relations are nothing but the de Broglie's conjecture in his attempt to explain the dual nature of matter as both particle and wave by relating the particle properties (energy-momentum) of the matter to the wave properties (angular frequency-wave number) of the matter. Yet, in contrast to our approach which is based on *linear wave equation*, de Broglie envisioned such soliton solution to be derived from a *nonlinear wave equation*.

Now, let us assume that the internal vibration resonates inside the spatio-temporal support. Namely, the wave number of the internal vibration are equal to the integer multiple of the wave number of the spatial part of quantum amplitude; and moreover, the angular frequency of the internal vibration is equal to the integer multiple of the angular frequency of the temporal part of the quantum amplitude:

$$\begin{aligned} k_{Bi} &= n_i k_{i_0}, \quad i = x, y, z, \\ \omega_B &= n_t \omega_0. \end{aligned} \quad (52)$$

where n_x, n_y, n_z, n_t are integer. Multiplying both sides of the above equations with Planck constant \hbar , one gets

$$\begin{aligned} p_i &= \hbar k_{Bi} = n_i \hbar k_{i_0}, \quad i = x, y, z, \\ E &= \hbar \omega_B = n_t \hbar \omega_0. \end{aligned} \quad (53)$$

The above obtained relations tells us that the energy-momentum are quantized into discrete values.

Further, let us proceed to discuss a special case when $n_x = n_y = n_z = n_t = 1$ to have

$$E = \hbar \omega_0, \quad \mathbf{p} = \hbar \mathbf{k}_{\mathbf{x}_0}. \quad (54)$$

Using the above relation between the energy-momentum of the soliton and the angular frequency-wave number of the spacetime quantum amplitude, then Eq. (33) translates into:

$$\frac{E^2}{c^2} - \mathbf{p}^2 = m^2 c^2. \quad (55)$$

This is the classical energy-momentum relation of special relativity. Since in general one can choose $\mathbf{k}_{\mathbf{x}_0}$ and ω_0 independently of \mathbf{p} and E , then the above result opens the possibility of the violation of classical energy-momentum relation provided that Eq. (54) is not satisfied.

It is then interesting to ask when the resonance of the internal spatio-temporal vibration inside the spacetime support occurs. To answer this, it is intuitive to learn from the classical wave phenomena. In this case, the resonance phenomena occurs if there is interaction and transfer of energy. One therefore might guess that the quantization of energy-momentum of Eqs. (53) occurs if there is interaction and transfer of energy-momentum. In other words, we expect that the energy-momentum can only be transferred from one matter to the other through interaction in discrete quantized value. For the case of light, this statement is but the Planck-Einstein conjecture which eventually gave birth to quantum theory. To discuss this issue in precise manner, one has to develop a theory describing interaction among particles.

D. Spacetime uncertainties

One of the important property of the spacetime soliton wave function we developed in the previous subsections is that it is broadened with finite width both in space and time axes. Namely, any particle should be considered as an extended object both in space and time. One is thus suggested to abandon the view of seeing a particle as a *dimensionless point* except for approximation to certain situation.

It is well-known that the assumption of point particle has led to many formal and physical difficulties in physics. It is for example argued as the origin of the divergence of calculations in the current version of quantum field theory. Another example is the famous paradox of self-interference in double slits experiment [3]. Assuming a point particle will force one to say an ambiguous sentence that *the particle is passing through both slits to interfere with itself at the screen*. Our soliton model of particle developed in this paper might then be considered as prospective candidate to address these important issues. It is thus imperative to understand how the width of the soliton wave function is related to other directly observable physical quantities.

First, let us discuss the spatial width of the soliton wave function defined by $\Delta_i = 2i_0$, $i = x, y, z$. Recalling the relation $k_{i_0}i_0 = \pi/2$, $i = x, y, z$ of Eq. (22), which is coming from

the boundary condition at the blowing-up point, one has

$$\Delta_i = \frac{\pi}{k_{i_0}}, \quad i = x, y, z. \quad (56)$$

Hence the width in the i -axis is proportional to the inverse of the i -part of the wave number of the spatial quantum amplitude. To see its relation with directly observable quantities, it is convenient to define the following quantity:

$$\begin{aligned} \Delta_{\mathbf{x}} &\equiv \frac{\Delta_x \Delta_y \Delta_z}{\sqrt{(\Delta_y)^2 (\Delta_z)^2 + (\Delta_x)^2 (\Delta_z)^2 + (\Delta_x)^2 (\Delta_y)^2}} \\ &= \frac{\pi}{\sqrt{k_{\mathbf{x}_0}^2}} = \frac{\pi \hbar}{\sqrt{\frac{\hbar^2 \omega_0^2}{c^2} - m^2 c^2}}. \end{aligned} \quad (57)$$

It depends on the angular frequency of the quantum amplitude and also on mass. In the case when the internal vibration resonates inside the spatio-temporal support in its first mode, the denominator is equal to the absolute value of the momentum of the particle to give:

$$\Delta_{\mathbf{x}} = \frac{\pi \hbar}{\sqrt{\frac{E^2}{c^2} - m^2 c^2}} = \frac{\pi \hbar}{\|\mathbf{p}\|}. \quad (58)$$

Next, let us discuss the temporal width of the soliton wave function defined similarly by: $\Delta_t \equiv 2t_0$. Again, using the relation $\omega_0 t_0 = \pi/2$ of Eq. (29), one gets

$$\Delta_t = \frac{\pi}{\omega_0} = \frac{\pi \hbar}{c \sqrt{\hbar^2 k_0^2 + m^2 c^2}}. \quad (59)$$

Similarly, notice that if the internal vibration beats inside the spatio-temporal support in its first mode, the denominator is just equal to the energy E to give us

$$\Delta_t = \frac{\pi \hbar}{E} \quad (60)$$

IV. SUPERPOSITION OF MASSES

Now let us proceed to discuss the implication of the linearity of the Schrödinger equation of Eq. (2). Recall that the soliton wave function given in Eq. (50) can be interpreted as the wave function of a single free particle with a given mass m , energy E and momentum $\mathbf{p} = \{p_x, p_y, p_z\}$. Since the Schrödinger equation of Eq. (2) is linear with respect to the wave function, then any superposition of the solutions of the type given in Eq. (50) will also satisfy the Schrödinger equation.

For example, let us assume that $\psi_i(q; \tau; \{m_i, E_i, \mathbf{p}_i\})$, $i = 1, 2$ takes the soliton form given in Eq (50) with the mass m_i , energy-momentum $\{E_i, \mathbf{p}_i\}$, $i = 1, 2$. Then, the following wave function which is obtained by superposing the two solitons wave functions:

$$\begin{aligned} \psi(q; \tau) = & A_1 \psi_1(q; \tau; \{m_1, E_1, \mathbf{p}_1\}) \\ & + A_2 \psi_2(q; \tau; \{m_2, E_2, \mathbf{p}_2\}), \end{aligned} \quad (61)$$

also satisfies the Schrödinger equation of Eq. (2). Here A_i , $i = 1, 2$ are two complex-valued constants which satisfy: $|A_1|^2 + |A_2|^2 = 1$. One can thus interpret each soliton term on the right hand side as describing a particle with various values of masses, energies and momenta.

Notice that at period of time when the spatio-temporal support of both solitons wave functions are overlapping, one will observe interference in space and time. Hence it is impossible to distinguish one soliton from the other. One therefore should consider both as a single particle. Now let us assume that $\mathbf{p}_1 = -\mathbf{p}_2$, namely each soliton is moving with direction opposite to the movement of the other. Then at sufficiently large time τ , the two solitons are no more overlapping in space so that one can already distinguish one from the other. Both might however still overlap in time domain. If further one chooses $E_1/m_1 \neq E_2/m_2$, then at sufficiently large proper time, both soliton will also separate in time axis.

Notice also that both solitons move independently from the other. One can thereby calculate the total mass, m_{12} , to obtain

$$m_{12} = \sqrt{\frac{-2\bar{U}}{c^2}} = m_1 + m_2. \quad (62)$$

This can be shown easily by choosing proper time τ so that both solitons are not overlapping in spacetime so that one has $\bar{U} = \bar{U}_1 + \bar{U}_2$, where \bar{U}_i is the average quantum potential of soliton $i = 1, 2$. Hence, the manifestly covariant Schrödinger equation of Eq. (2) can be seen as field theory in which a particle of mass m_{12} can break into two particles of masses m_1 and m_2 so that $m_1 + m_2 = m_{12}$. Each particle moves independently with momentum \mathbf{p}_i and possessing energy E_i , where $i = 1, 2$. Moreover, since energy and momentum must be conserved then one can define the total energy-momentum as $E_{12} = E_1 + E_2$ and $\mathbf{p}_{12} = \mathbf{p}_1 + \mathbf{p}_2$. Conversely, since the Schrödinger equation of Eq. (2) is time reversal, then by reversing the evolution of affine parameter, the initially separated two solitons each describing a particle of mass

m_i , $i = 1, 2$, can merge into a single particle with mass $m_{12} = m_1 + m_2$. Needless to say, extension into the superposition of more than two solitons are straight forward.

V. CONCLUSION AND DISCUSSION

We have thus shown that first, the manifestly covariant mass-less Schrödinger equation of Eq. (2) possesses a class of non-dispersive soliton solutions with finite-size spatio-temporal support. The quantum amplitude inside the support satisfies Klein-Gordon equation with finite emergent mass. We finally came to give an interpretation to the soliton solution as describing a single particle with finite mass, energy and momentum. Moreover, we showed that the soliton solution possesses internal spatio-temporal vibration with angular frequency and wave number which are determined by the energy-momentum of the particle as exactly conjectured by de Broglie while proposing the solution for the problem of duality of matter as both particle and wave. However, in contrast to his envision of *nonlinear wave equation* assuming soliton solution, we showed that our soliton wave function is a solution of a *linear wave equation*. We argued that the fact that the soliton wave function has finite extension in spacetime might give the key to the problematic divergence problem encountered in the current version of quantum field theory and also to the paradox of particle self-interference in double slits experiment.

We showed that if the internal vibration resonates inside the spatio-temporal support, then one can show that the energy and momentum are discretized into packets as assumed by Planck and Einstein. We further suggested that this situation appears if there is interaction among matters which induces transfer of energy. We also showed that the classical energy-momentum relation is recovered when the internal vibration resonates inside the spatio-temporal support in its first excited mode. This opens the possibility of violating the energy-momentum relation if one moves away from the first resonance mode.

Next, the linearity of the Schrödinger equation allows for superposition of such solutions to further comprise a class of many solitons solutions. Each term of the superposition then describes a single particle with a given mass, energy and momentum, moving independently of the other. One can thus consider the wave function as a real physical field and regards the Schrödinger equation of Eq. (2) as a field theory describing multi-particles systems. In this theory, things thus live in the ordinary three dimensional space rather than in configuration

space.

There are at least three other interesting issues suggest further exploration. First, we have identified soliton as particle by attributing to it mass and energy-momentum. It is then natural to ask: what about spin? Second, since the soliton wave function is broadened also in time axis, then it might be possible to see interference in the time domain [25]. Finally third, our theory allows us to construct a superposition of two solitons which initially interferes each other but then is separated from each other as time goes. It is then interesting to employ this idea to re-think the EPR thought experiment.

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- [1] Roger Penrose, *The large, the small and human mind* (Cambridge University Press, Cambridge, 2000).
 - [2] C. J. Isham, *Lectures On Quantum Theory: Mathematical and Structural Foundation* (Imperial College Press, London, 1995).
 - [3] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 2004).
 - [4] D. Bohm and B. J. Hiley, *The Undivided Universe: An ontological interpretation of quantum theory* (Routledge, London, 1993).
 - [5] M. Born Z. Phys. **37**, 863 (1926).
 - [6] P. A. M. Dirac, *The Principle of Quantum Mechanics* (Clarendon Press, UK, 1981).
 - [7] John von Neumann, *Mathematical Foundation of Quantum Mechanics* (Princeton University Press, Princeton, 1996).
 - [8] D. Bohm, Phys. Rev. **85**, 166 (1952); **85**, 180 (1952); **89**, 458 (1953).
 - [9] H. Everett, Rev. Mod. Phys. **29**, 454 (1957).
 - [10] B. S. DeWitt and N. Graham, *The Many-Worlds Interpretation of Quantum Mechanics* (Princeton University Press, Princeton, 1973).
 - [11] G. C. Ghirardi, A. Rimini, and T. Weber, Phys. Rev. D **34**, 470 (1986).

- [12] L. de Broglie, *Une tentative d'interpretation causale et nonlineaire de la mecanique ondulatoire* (Gauthier-Villars, Paris, 1956).
- [13] L. de Broglie, *Heisenberg's Uncertainties and the Probabilistic Interpretation of Wave Mechanics* (Kluwer Academic Publisher, London, 1990).
- [14] W. Pauli, *General principles of quantum mechanics* (Springer, Berlin, 1980).
- [15] N. Curfaro-Petroni *et. al.*, Phys. Rev. D **32**, 1375 (1985).
- [16] J. P. Vigier, Astron. Nachr. **303**, 1 (1982).
- [17] J. P. Vigier, Phys. Lett. A **135**, 99 (1989).
- [18] L. Mackinnon, Found. Phys. **8**, 157 (1978).
- [19] A. O. Barut, Phys. Lett. A **143**, 349 (1990).
- [20] Y. Nambu, Prog. Theor. Phys. **5**, 82 (1950).
- [21] A. Kyprianidis, Phys. Rep. **155**, 1 (1987).
- [22] E. Madelung, Zeits. F. Phys. **40**, 332 (1926).
- [23] Agung Budiyono and Ken Umeno, Phys. Rev. A **79**, 042104 (2009).
- [24] S. Alinhac, *Blowup for Nonlinear Hyperbolic Equations* (Birkhäuser, Basel, 1995).
- [25] F. Lindner, M. G. Schätzel, H. Walther, A. Baltuska, E. Goulielmakis, F. Krausz, D.B. Milosevic, D. Bauer, W. Becker and G.G. Paulus, Phys. Rev. Lett. **95**, 040401 (2005).